

Geometric Algebra conference: Fields Institute

July 7, 2015

On Quantum Groups assoc. to non-Noetherian regular algs of dim 2

joint work w/ Xingting Wang

arXiv: 1503.09185

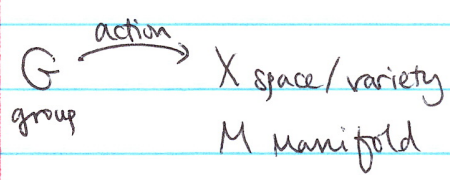
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Note to self: can probably skip some material, depends on Toby's course & Raedschelders' talk

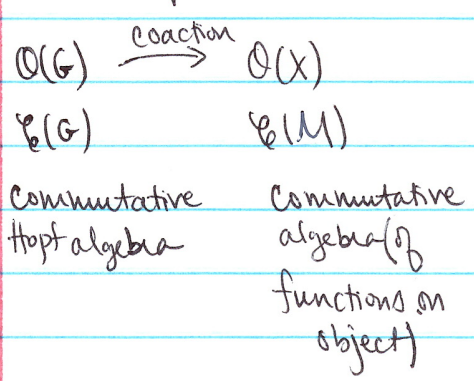
Motivation:

Classic Symmetry

Symmetry of geometric objects is governed by group actions



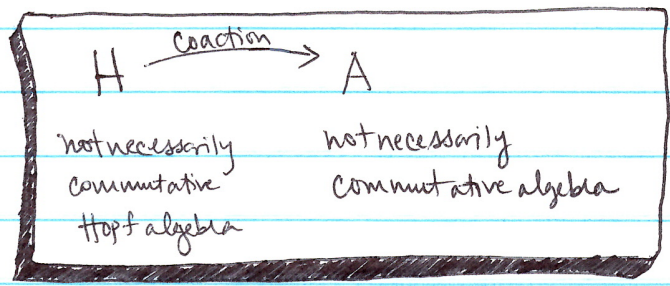
dualize



Quantum Symmetry

Can't see object

dualizing works well



— what we aim to study —

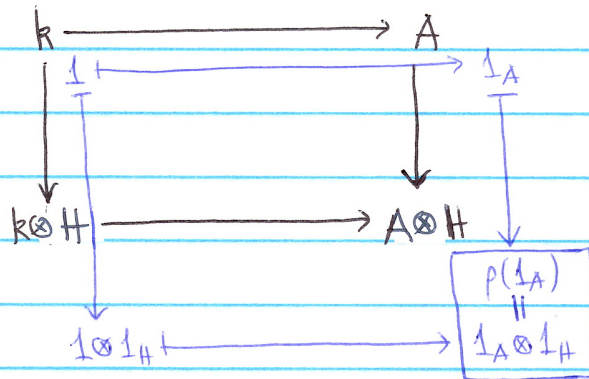
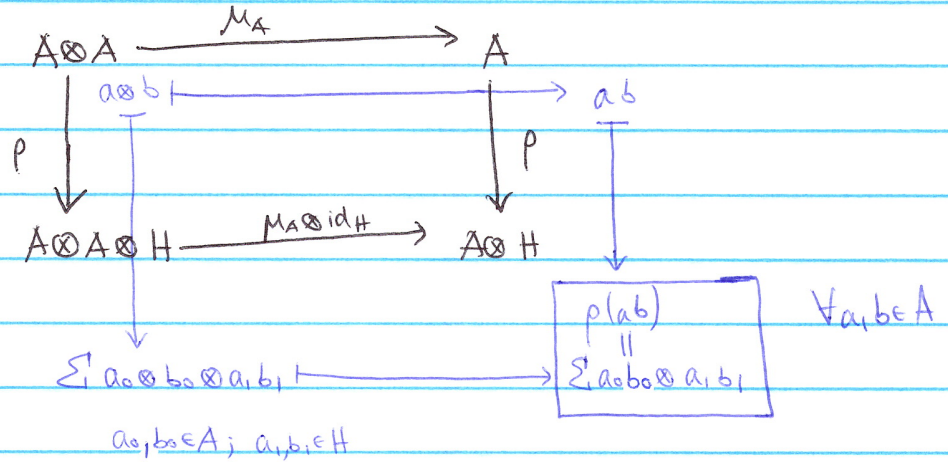
A Hopf algebra (or quantum group)  $H$  is

- an associative algebra  $[(H, \mu, \eta)$  with  $\mu \circ (\text{id} \otimes \mu) = \mu \circ (\mu \otimes \text{id})$  mult. unit.]
- a coassociative coalgebra  $[(H, \Delta, \epsilon)$  with  $(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta$  co-mult. co-unit.]
- with antipode  $S: H \rightarrow H$  "inverse"

... subject to compatibility conditions.

$H$  coacts on an algebra  $A$  if  $A$  is an  $H$ -comodule algebra:

- $A$  is a  $H$ -comodule with structure map  $\rho: A \rightarrow A \otimes H$   
 $a \mapsto \sum a_i \otimes a_i$
- $\mu_A$  (mult) and  $\nu_A$  (unit) are  $H$ -comodule maps

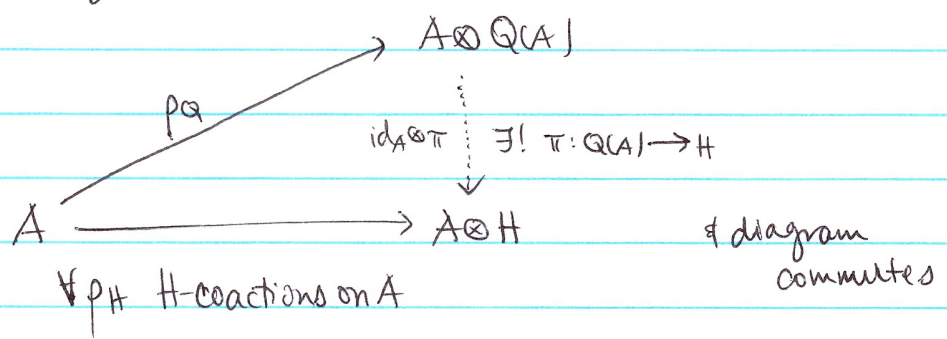


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In other words,  $A$  is an algebra in the category of  $H$ -comodules

Given an graded algebra  $A$  gen. in deg. 1  $\xrightarrow{\text{universal quantum group } Q(A)}$  linear  $Q(A)(GL)$  that coact on  $A$  preserving the grading of  $A$ .

if  $Q(A)$  is a Hopf algebra so that:



$Q(A)$  is tough to study, so let's focus on a special case  
 $Q_A(GL)$ : introduced by Manin for graded quadratic algebras (1988)

Examples

	$A$	$Q_A^c(GL)$ with central "determinant" (discussed later)	$Q_A(SL)$ with trivial "determinant"
	$k\langle u, v \rangle$	$Q(GL_2(k))$	$Q(SL_2(k))$
$q \in k^\times$	(1-parameter deformation) $k_q\langle u, v \rangle = \frac{k\langle u, v \rangle}{(uv - quv)}$	$Q_q(GL_2(k))$	$Q_q(SL_2(k))$
	(Jordanian deformation) $k_J\langle u, v \rangle = \frac{k\langle u, v \rangle}{(uv - vu - v^2)}$	$Q_J(GL_2(k))$	$Q_J(SL_2(k))$
	familiar quantum algebras - These are the <u>Noetherian Artin-Schelter reg. algs</u> of dimension 2 (cf. Toby's course)	familiar Hopf algebras (presentation later...)	discussed later

(Just in case this is not yet defined in Toby's course)

An Artin-Schelter (AS) regular algebra (of dimension  $d$ )

is a connected graded  $k$ -algebra  $A = k \oplus A_1 \oplus A_2 \oplus \dots$   
 of global dimension  $d < \infty$  that is AS-Gorenstein [ $\text{Ext}_A^i(k, A) = \delta_{i,d} k$ ]

↗ sharing nice homological properties with  $k[u_1, u_2, \dots, u_d]$

In fact, all of the (Hopf) algebras above are AS-regular

& Noetherian, domain, have poly'l growth ) nice ring-theoretic properties

**PHILOSOPHY**

THE UNIV QUANTUM LINEAR GROUPS  $Q_A(GL)$  (with additional conditions?)

SHOULD SHARE THE SAME RING-THEORETIC

& HOMOLOGICAL PROPERTIES OF THE COMODULE ALGEBRA  $A$

↗

verified for many classes of Noetherian AS regular algebras  $A$   
 (of dimension 2, skew poly'l rings, etc.)

We investigate the case when  $A$  is not necessarily Noetherian.  
 First, with global dimension 2:

[Zhang] The AS regular algs of global dimension 2 are

$$A(\mathbb{E}) = A(n, \mathbb{E}) = \langle k\langle v_1, \dots, v_n \rangle / \left( \sum_{i=1}^n e_{ij} v_i v_j \right) \rangle, \quad \mathbb{E} = (e_{ij}) \in GL_n(k)$$

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$$\text{eg } A\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = k[u, v] \quad A\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = k_0[u, v] \quad A\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right) = k_0[u, v]$$

We consider Manin's QGS  $\mathcal{O}_{A(\mathbb{k})}(GL)$  in two special cases:

with central  $\left\{ \begin{array}{l} \text{quantum determinant} \\ \text{homological codet.} \\ \text{"determinant"} \end{array} \right.$

with squared antipode of finite order

$\mathcal{O}(\mathbb{k}^+; \mathbb{k}) \equiv \mathcal{O}_{A(\mathbb{k})}^c(GL)$

a little tougher to compute  $\mathcal{O}_{A(\mathbb{k})}(GL/S^{2m})$

(see next page)

which includes trivial determinant

which includes the involution case

$\mathcal{O}(\mathbb{k}^-) \equiv \mathcal{O}_{A(\mathbb{k})}(SL)$

$\mathcal{O}(\mathbb{k}^+; \mathbb{k}^-) \equiv \mathcal{O}_{A(\mathbb{k})}(GL/S^2)$

eg.  $\curvearrowright$  is the univ. Hopf alg that coacts on  $A(\mathbb{k})$  with trivial det.

eg.  $\curvearrowright$  is the univ. involutory Hopf alg. that coacts on  $A(\mathbb{k})$

Usually the determinant is tricky to compute for Hopf coactions, but it's easy for Hopf coactions on  $A(\mathbb{k})$ :

[Chen-Kirkman-W-Zhang] The determinant of an H-coaction on  $A(\mathbb{k}) = \mathbb{k}\langle V \rangle / \langle r \rangle$  is the (group-like) element  $D \in H$  so that  $\rho: A \rightarrow A \otimes H$   
 $r \mapsto r \otimes D^{-1}$

$D$  is central of  $[D, h] = 0 \forall h \in H$ ,  $D$  is trivial if equal to  $1_H$ .

Advantages to special cases:

- ① There are "homological identities" that relate the two cases - see work of Chen-W-Zhang, Reyes-Rogalski-Zhang [1210.0432: Theorem 0.1] (conj by  $D$ )  $\circ S^2 =$  (conj by transpose of Nakayama autom)  $\left[ \begin{array}{l} \text{trivial when } A \text{ is cty} \end{array} \right]$

- ② Have a "nice" presentation of  $\mathcal{O}_{A(\mathbb{k})}^*(*)$  in these cases: (more amenable to computations)

[Mrozinski (2011)] Take  $E, F \in GL_n(k)$ , for  $n \geq 2$

Take  $\mathcal{G}(E, F)$  to be the Hopf algebra generated by  $A = (a_{ij}), D^{\pm 1}$   
 $1 \leq i, j \leq n$ .  
with relations:

$$AE^{-1}A^T E = DI = F A^T F^{-1} A, \quad DD^{-1} = D^{-1}D = 1$$

(See paper for coalgebra structure & antipode).

Get that  $\mathcal{G}(E^{-1}, F)$  coacts on  $A(E)$  with determinant  $D^{-1}$

[Dubois-Violette and Lauer (1990)]  $B(E) = \mathcal{G}(E, E^{-1}) / (D-1)$   
(Notice the dates; this is <sup>(univ.)</sup> "quantum group of a nondeq. bilinear form" <sup>(that preserves)</sup>)

Get that  $B(E^{-1})$  coacts on  $A(E)$  with determinant 1.

Now we show that (see previous page)

Computed	$Q_{A(E)}^c(GL)$	$Q_{A(E)}(SL)$	$Q_{A(E)}(GL/S_{2m})$
	- explicitly for $E \in GL_2(k)$ at end of Section 2 of paper.		
ex. for $A = k\langle u, v \rangle$ :	$Q_q(GL_2(k))$	$Q_q(SL_2(k))$	$Q_{q, q^{-1}}(GL_2(k))$ for all $m \geq 1$ .
			(Takeuchi's two parameter deformation of $Q(GL_2(k))$ )

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Now we have the following results and questions!

Results & Questions

Comments

$\mathbb{k} \in GL_n(\mathbb{k})$

Properties of  $A = A(n, \mathbb{k})$

Our Results

of  $\mathbb{Q}_A(SL)$

Questions

of  $\mathbb{Q}_A^c(GL)$  & of  $\mathbb{Q}_A(GL/S^2)$

HOMOLOGICAL

- AS reg.
  - global dimension 2
  - AS Gorenstein
- skew Calabi-Yau
  - homologically smooth of dim 2
  - A has min & proj resolution in  $A^e$  mod of length 2.
  - rigid Gorenstein
  - Alg. autom  $\psi$  of A s.t. that
    - $Ext_{A^e}^i(A, A^e) \cong d_{i,2} \psi A^{\pm 1}$
    - $\cong A$ -bimodules
    - $\psi =$  Nakayama Autom.
- Calabi-Yau [ $\psi$  is inner]
  - $\Leftrightarrow$  never,  $\mathbb{k}$  skew symmetric
- Koszul

gldim 3 } AS regular  
AS Goren. } Hopf alg.

homol. smooth of dimension 3 } skew Calabi-Yau  
rigid Gorenstein

$\mathbb{k}$  symmetric or skew-sym  
 $\Rightarrow \mathbb{B}(\mathbb{k}^{-1})$  w/ Cy and involutory

associated graded Koszul ???



Results based on work of Bichon: he produced a yetter-Drinfeld resolution of the counit of  $\mathbb{B}(\mathbb{k})$

probably need an equally nice resolution of the counit of  $\mathbb{B}(\mathbb{k}, \mathbb{k})$  to get started ....

RING-THEORETIC

- Noetherian  $\Leftrightarrow n=2$
- finite Gelfand-Kirillov dim.  $\Leftrightarrow n=2$   
(poly & growth)
- domain
- graded coherent
- Hilb. series  $\frac{1}{1-nt+t^2}$

Noetherian  $\Leftrightarrow n=2$

finite GKdim  $\Leftrightarrow n=2$

?

?(coherent)

?

Noetherian  $\Leftrightarrow n=2$

finite GKdim  $\Leftrightarrow n=2$

?

?

cooking up a monomial basis is very tough! Depends on variable ordering & choice of  $\mathbb{k}$ !

... speaking of choice of  $\mathbb{E}$ .

We are able to do computations because of the following results:

[Zhang, Bichen, Kozinski] For  $P \in GL_n(k)$ :

- ①  $A(\mathbb{E}) \simeq A(P^T \mathbb{E} P)$  as algebras
- ②  $\mathcal{B}(\mathbb{E}) \simeq \mathcal{B}(P^T \mathbb{E} P)$  as Hopf algebras.
- ③ If  $\mathbb{E}^T P^T \mathbb{E} P = \lambda I$  for some  $\lambda \in k^\times$ , then  $\mathcal{B}(P \mathbb{E} P^{-1}) \simeq \mathcal{B}(P^T \mathbb{E} P, P^{-1} \mathbb{E} (P^{-1})^T) \simeq \mathcal{B}(P^T \mathbb{E}^{-1} P, P^{-1} \mathbb{E}^{-1} (P^{-1})^T)$  as Hopf algebras.

That is, can replace  $\mathbb{E}$  with a matrix congruent to  $\mathbb{E}$

[Horn-Degeichak] Each  $\mathbb{E} \in GL_n(k)$  is congruent to a direct sum, uniquely determined up to permutation of summands, of the following matrices

$$\begin{aligned}
 \mathbb{J}_n &= \begin{pmatrix} \bigcirc & & & \\ & \ddots & & \\ & & (-1)^{n+1} & \\ & & & (-1)^n & \\ & 1 & & & \\ & & 1 & & \\ & & & & \bigcirc \end{pmatrix} \quad \text{"Jordan-type"} \\
 \mathbb{D}_2(q) &= \begin{pmatrix} \bigcirc & 1 & \dots & 0 \\ & \ddots & & \\ q & & & \\ & \ddots & & \\ 0 & & & q \\ & & & & \bigcirc \end{pmatrix} \quad \begin{array}{l} 2r=n \\ q \in k \setminus \{0, (-1)^{r+1}q\} \end{array} \\
 & \quad \text{"q-type"}
 \end{aligned}$$

Eg.  $n=2 \rightsquigarrow \mathbb{E}$  congruent to

$$\mathbb{J}_1 \oplus \mathbb{J}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbb{J}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \mathbb{D}_2(q) = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \quad q \neq 0, 1$$

[4:53]

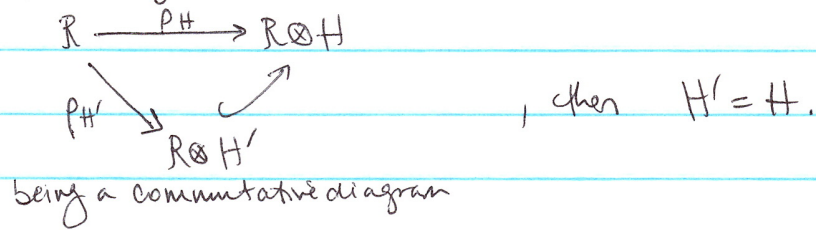
$\rightsquigarrow k[i, v]$ ,  $k\sigma[u, v]$ ,  $k-q[u, v]$  are the Admg algs up to isomorphism.



On the other hand, have results on when Hopf coactions on AS regular algs factor through coactions of cocommutative Hopf algebras ---

An H-coaction on an algebra R is inner faithful if the coaction does not factor through the coaction of a proper Hopf subalg  $H' \subsetneq H$ .

That is, if  $\exists$  Hopf subalgebra  $H' \subsetneq H$  with



Theorem  $R = \begin{cases} N\text{-Koszul} \\ \text{Noetherian} \end{cases}$  AS regular alg., with  $\dim_k R_1 = n$ .  
 $H =$  Hopf alg with antipode of finite order coacting on R inner faithfully with determinant D.

$M =$  matrix corresponding to Nakayama automorphism of R (ie.  $M = \Pi$  if R is cy)  
 $\mathcal{C}(M) = \{P \in \text{Mat}_n(k) \mid PM = MP\}$  (centralizer of M), subalg. of  $\text{Mat}_n(k)$   
 $\mathcal{C}_p(M) = \bigcup_{i \geq 1} \mathcal{C}(M^i)$  (power centralizer of M)

*Admittedly we still need to understand this further*

If either ①  $\mathcal{C}(M)$  commutative, H involutory, D central, or ②  $\mathcal{C}_p(M)$  commutative,  $D^m$  central for some  $m \geq 1$ , then H is cocommutative

Corollary Take  $R = A(\mathbb{E})$  with  $\mathbb{E}$  generic (as specified in paper), H as above. If ① H involutory, D central, or ②  $D^m$  central for some  $m \geq 1$ , then H is cocom.

*when  $R = A(\mathbb{E})$   $\mathbb{E} \Leftrightarrow \mathbb{E}$  being generic.*

Next: coactions on higher dimensional regular algebras. X. Wang & A. Chirvasiu