Geometric Algebra conference: Fields Institute

July 7, 2015

On Quantum Groups assoc. to non-Noetherian regular algs of dim 2

joint work w/ Xingting Wang arXiv: 1503.09185.

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Note to self: can probably skip some material, depends on Toby's course & Raedschelders' talk

Motivation:

Classic Symmetry

(can see these)

symmetry of geometric objects is governed by group actions

$\underset{\text{group}}{G} \xrightarrow{\text{action}} X$ space/variety
M manifold

↓ dualize

$\mathcal{O}(G) \xrightarrow{\text{coaction}} \mathcal{O}(X)$

$\mathcal{C}(G)$ $\mathcal{C}(M)$

commutative Hopf algebra

commutative algebra (of functions on object)

Quantum Symmetry

Can't see object

↓ dualizing works well

$H \xrightarrow{\text{coaction}} A$

not necessarily commutative Hopf algebra

not necessarily commutative algebra

— what we aim to study —

A **Hopf algebra** (or **quantum group**) H is

- an associative algebra $[(H, \underset{\text{multip.}}{\mu}, \underset{\text{unit}}{u})$ with $\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id)]$

- a coassociative coalgebra $[(H, \underset{\text{comult.}}{\Delta}, \underset{\text{counit}}{\varepsilon})$ with $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta]$

- with antipode $\underset{\text{"inverse"}}{S}: H \to H$

... subject to compatibility conditions.

H $\boxed{\text{coacts}}$ on an algebra A if A is an $\boxed{H\text{-comodule algebra:}}$:

- $\boxed{A \text{ is a } H\text{-comodule}}$ with structure map $\rho: A \to A \otimes H$, $a \mapsto \Sigma\, a_0 \otimes a_1$
- μ_A (mult) and u_A (unit) are H-comodule maps

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu_A"] \arrow[d, "\rho"'] & A \arrow[d, "\rho"] \\ A \otimes A \otimes H \arrow[r, "\mu_A \otimes id_H"] & A \otimes H \end{tikzcd}$$

$$\begin{tikzcd} a \otimes b \arrow[r, mapsto] \arrow[d, mapsto] & ab \arrow[d, mapsto] \\ \Sigma\, a_0 \otimes b_0 \otimes a_1 b_1 \arrow[r, mapsto] & \boxed{\begin{matrix} \rho(ab) \\ \parallel \\ \Sigma\, a_0 b_0 \otimes a_1 b_1 \end{matrix}} \end{tikzcd}$$

$\forall a, b \in A$

$a_0, b_0 \in A$; $a_1, b_1 \in H$

$$\begin{tikzcd} k \arrow[r] \arrow[d] & A \arrow[d] \\ k \otimes H \arrow[r] & A \otimes H \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r, mapsto] \arrow[d, mapsto] & 1_A \arrow[d, mapsto] \\ 1 \otimes 1_H \arrow[r, mapsto] & \boxed{\begin{matrix} \rho(1_A) \\ \parallel \\ 1_A \otimes 1_H \end{matrix}} \end{tikzcd}$$

$\boxed{4{:}15}$ In other words, A is an algebra in the category of H-comodules

Given an graded algebra A, gen. in deg. 1, say $\exists$ $\boxed{\text{universal linear quantum group } \mathcal{Q}(A)}$ $\mathcal{O}_A(GL)$ that coact on A preserving the grading of A.

if $\mathcal{Q}(A)$ is a Hopf algebra so that:

$$\begin{tikzcd}
 & & A \otimes \mathcal{Q}(A) \arrow[d, dotted, "id_A \otimes \pi"'] \\
A \arrow[urr, "\rho_Q"] \arrow[rr] & & A \otimes H
\end{tikzcd}$$

$\exists!\ \pi: \mathcal{Q}(A) \to H$

$\forall \rho_H$ H-coactions on A & diagram commutes

$\mathcal{Q}(A)$ is tough to study, so let's focus on a $\boxed{\text{special case}}$

$\mathcal{O}_A(GL)$: introduced by Manin for graded quadratic algebras (1988)

$\boxed{\text{Examples}}$

	A	$\mathcal{O}^c_A(GL)$ with central "determinant" (discussed later)	$\mathcal{O}_A(SL)$ with trivial "determinant"
	$k[u,v]$	$\mathcal{O}(GL_2(k))$	$\mathcal{O}(SL_2(k))$
$q \in k^\times$ (1-parameter deformation)	$k_q[u,v] = \frac{k\langle u,v\rangle}{(uv - qvu)}$	$\mathcal{O}_q(GL_2(k))$	$\mathcal{O}_q(SL_2(k))$
(Jordanian deformation)	$k_J[u,v] = \frac{k\langle u,v\rangle}{(uv - vu - v^2)}$	$\mathcal{O}_J(GL_2(k))$	$\mathcal{O}_J(SL_2(k))$

algebraic groups that act on $k[u,v]$

familiar quantum algebras — These are the Noetherian Artin-Schelter reg. algs of dimension 2 (cf. Toby's course)

familiar Hopf algebras (presentation later...)

discussed later

(Just in case this is not yet defined in Toby's course)

An $\boxed{\text{Artin-Schelter (AS) regular algebra (of dimension } d\text{)}}$ is a connected graded k-algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ of global dimension $d < \infty$ that is AS-Gorenstein $[\mathrm{Ext}^i_A(k,A) = \delta_{i,d}k]$

sharing nice homological properties with $k[u_1, u_2, \ldots, u_d]$

In fact, all of the (Hopf) algebras above are AS-regular & Noetherian, domain, have poly'l growth) nice ring-theoretic properties

PHILOSOPHY: THE UNIV QUANTUM LINEAR GROUPS $\mathcal{O}_A(GL)$ (with additional conditions?) SHOULD SHARE THE SAME RING-THEORETIC & HOMOLOGICAL PROPERTIES OF THE COMODULE ALGEBRA A

Verified for many classes of Noetherian AS regular algebras A (of dimension 2, skew poly'l rings, etc.)

We investigate the case when A is not necessarily Noetherian.

First, with global dimension 2:

[Zhang] The AS regular algs of global dimension 2 are

$$A(\mathbb{E}) = A(n, \mathbb{E}) = k\langle v_1, \ldots, v_n\rangle / \left(\sum_{i=1}^{n} e_{ij} v_i v_j\right), \quad \mathbb{E} = (e_{ij}) \in GL_n(k)$$

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eg. $A\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = k[u,v]$, $A\left(\begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}\right) = k_q[u,v]$, $A\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right) = k_J[u,v]$

We consider Manin's QGs $\mathcal{O}_{A(\mathbb{E})}(GL)$ in two special cases:

with central { quantum determinant / homological codet. "determinant" }

with squared antipode of finite order

$\mathcal{D}(\mathbb{E}^{-1};\mathbb{E}) \cong \boxed{\mathcal{O}^{c}_{A(\mathbb{E})}(GL)}$

a little tougher to compute

$\boxed{\mathcal{O}_{A(\mathbb{E})}(GL/S^{2m})}$

(see next page)

which includes trivial determinant

$\mathcal{B}(\mathbb{E}^{-1}) \cong \boxed{\mathcal{O}_{A(\mathbb{E})}(SL)}$

e.g. is the univ. Hopf alg that coacts on $A(\mathbb{E})$ with trivial det.

which includes the involutory case

$\mathcal{D}(\mathbb{E}^{-1};\mathbb{E}^{T}) \cong \boxed{\mathcal{O}_{A(\mathbb{E})}(GL/S^{2})}$

e.g. is the univ. involutory Hopf alg. that coacts on $A(\mathbb{E})$

Usually the determinant is tricky to compute for Hopf coactions,

But it's easy for Hopf coactions on $A(\mathbb{E})$:

[Chan-Kirkman-W-Zhang] The determinant of an H-coaction on $A(\mathbb{E}) = k\langle V\rangle/(r)$ is the (grouplike) element $D \in H$ so that $\rho: A \to A \otimes H$, $r \mapsto r \otimes D^{-1}$

D is central if $[D,h]=0$ $\forall h \in H$, D is trivial if equal to 1_H.

Advantages to special cases:

① There are "homological identities" that relate the two cases

See work of Chan-W-Zhang, Reyes-Rogalski-Zhang.

[1210.6432: Theorem 0.1 $(\text{conj by } D) \circ S^2 = (\text{conj by transpose of Nakay. autom})$]

trivial when A is CY

② Have a "nice" presentation of $\mathcal{O}^{*}_{A(\mathbb{E})}(*)$ in these cases:

(more amenable to computations)

[Mrozinski (2014)] Take $\mathbb{E}, \mathbb{F} \in GL_n(k)$, for $n \geq 2$

Take $\boxed{\mathcal{G}(\mathbb{E}, \mathbb{F})}$ to be the Hopf algebra generated by $\mathbb{A} = (a_{ij})_{1 \leq i,j \leq n}, D^{\pm 1}$ with relations:

$$\mathbb{A}\mathbb{E}^{-1}\mathbb{A}^T\mathbb{E} = D\mathbb{I} = \mathbb{F}\mathbb{A}^T\mathbb{F}^{-1}\mathbb{A}, \quad DD^{-1} = D^{-1}D = 1$$

(see paper for coalgebra structure & antipode).

Get that $\mathcal{G}(\mathbb{E}^{-1}, \mathbb{F})$ coacts on $A(\mathbb{E})$ with determinant D^{-1}

[Dubois-Violette and Launer (1990)] $\mathcal{B}(\mathbb{E}) = \mathcal{G}(\mathbb{E}, \mathbb{E}^{-1}) / (D-1)$

(Notice the dates; this is (univ.) "quantum group (that preserves) of a nondeg. bilinear form")

Get that $\mathcal{B}(\mathbb{E}^{-1})$ coacts on $A(\mathbb{E})$ with determinant 1.

Now we show that (see previous page)

Computed $\mathcal{O}^c_{A(\mathbb{E})}(GL)$ $\quad \mathcal{O}_{A(\mathbb{E})}(SL)$ $\quad \mathcal{O}_{A(\mathbb{E})}(GL/\mathcal{S}_{2m})$

explicitly for $\mathbb{E} \in GL_2(k)$ at end of Section 2 of paper.

ex. for $A = k_q[u,v]$: $\mathcal{O}_q(GL_2(k))$ $\quad \mathcal{O}_q(SL_2(k))$ $\quad \mathcal{O}_{q,q^{-1}}(GL_2(k))$ for all $m \geq 1$.

(Takeuchi's two parameter deformation of $\mathcal{O}(GL_2(k))$)

4:40 Now we have the following results and questions!

Results & Questions

Comments

$\mathbb{E} \in GL_n(k)$

	Properties of $A = A(n, \mathbb{E})$	Our Results of $\mathcal{O}_A(SL)$	Questions of $\mathcal{O}_A^c(GL)$ & of $\mathcal{O}_A(GL/S^2)$
		not graded	
HOMOLOGICAL	AS reg: global dimension 2; AS Gorenstein	gldim 3, AS Goren.] AS regular Hopf alg.	?
	skew Calabi-Yau: homologically smooth of dim 2 [A has min'l proj resolution in A^e-mod of length 2.]; rigid Gorenstein [$\exists$ alg. autom ν of A so that $\mathrm{Ext}^i_{A^e}(A, A^e) \simeq \delta_{i,2}\, {}^{\nu}A^1$ as A-bimodules]; ν = Nakayama Autom	homol. smooth of dimension 3, rigid Gorenstein] skew Calabi-Yau	
	Calabi Yau [ν is inner] $\Leftrightarrow$ never, $\mathbb{E}$ skew symmetric	$\mathbb{E}$ symmetric or skew-sym $\Rightarrow$ $\mathcal{B}(\mathbb{E}^{-1})$ is CY and involutory	Results based on work of Bichon: he produced a Yetter-Drinfeld resolution of the counit of $\mathcal{B}(\mathbb{E})$
	Koszul	associated graded Koszul ???	probably need an equally nice resolution of the counit of $\mathcal{G}(\mathbb{E}, \mathbb{F})$ to get started ....
RING-THEORETIC	Noetherian $\Leftrightarrow$ $n=2$	Noetherian $\Leftrightarrow$ $n=2$	Noetherian $\Leftrightarrow$ $n=2$
		$\Leftarrow$ by brute force; $\Rightarrow$ show special Hopf quot. is not Noeth. (∞ GKdim)	
	finite Gelfand-Kirillov dim. (poly'l growth) $\Leftrightarrow$ $n=2$	finite GKdim $\Leftrightarrow$ $n=2$	finite GKdim $\Leftrightarrow$ $n=2$
	domain	?	?
	graded coherent	? (coherent)	?
	Hilb. series $\frac{1}{1-nt+t^2}$	?	?

cooking up a monomial basis is very tough! Depends on variable ordering & choice of $\mathbb{E}$!

... speaking of choice of $\mathbb{E}$.

We are able to do computations because of the following results:

[Zhang, Bichon, Mrozinski] For $\mathbb{P} \in GL_n(k)$:

① $A(\mathbb{E}) \simeq A(\mathbb{P}^T \mathbb{E} \mathbb{P})$ as algebras

② $\mathcal{B}(\mathbb{E}) \simeq \mathcal{B}(\mathbb{P}^T \mathbb{E} \mathbb{P})$ as Hopf algebras.

③ If $\mathbb{E}^T \mathbb{F}^T \mathbb{E} \mathbb{F} = \lambda \mathbb{I}$ for some $\lambda \in k^\times$, then

$$\mathcal{G}(\mathbb{F}, \mathbb{E}) \simeq \mathcal{G}(\mathbb{P}^T \mathbb{F} \mathbb{P}, \mathbb{P}^{-1} \mathbb{E} (\mathbb{P}^{-1})^T) \simeq \mathcal{G}(\mathbb{P}^T \mathbb{E}^{-1} \mathbb{P}, \mathbb{P}^{-1} \mathbb{F}^{-1} (\mathbb{P}^{-1})^T)$$

as Hopf algebras.

That is, can replace $\mathbb{E}$ with a matrix <u>congruent</u> to $\mathbb{E}$

[Horn-Sergeichuk] Each $\mathbb{E} \in GL_n(k)$ is congruent to a direct sum, uniquely determined up to permutation of summands, of the following matrices

$$\mathbb{J}_n = \begin{pmatrix} & 0 & & & (-1)^{n+1} \\ & & & \iddots & (-1)^n \\ & & \iddots & \iddots & \\ & -1 & -1 & & 0 \\ 1 & 1 & & & \end{pmatrix} \quad \& \quad \mathbb{D}_{2r}(q) = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline \begin{matrix} q & 1 & 0 \\ & \ddots & 1 \\ 0 & & q \end{matrix} & 0 \end{array}\right)$$

$2r = n$

"Jordan type" $\qquad q \in k \setminus \{0, (-1)^{r+1}\}$

"q-type"

Eg. $n=2 \rightsquigarrow \mathbb{E}$ congruent to

$$\mathbb{J}_1 \oplus \mathbb{J}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbb{J}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \qquad \mathbb{D}_2(q) = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$$

$q \neq 0, 1$

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$\rightsquigarrow \quad k_{-1}[u,v] \quad , \quad k_0[u,v] \quad , \quad k_{-q}[u,v]$ are the AS reg algs up to isomorphism.

On the other hand, have results on when Hopf coactions on AS regular algs factor through coactions of cocommutative Hopf algebras ...

An H-coaction on an algebra R is $\boxed{\text{inner faithful}}$ if the coaction does not factor through the coaction of a proper Hopf subalg $H' \subsetneq H$.

That is, if $\exists$ Hopf subalgebra $H' \subseteq H$ with

$$\begin{tikzcd} R \arrow[rr, "\rho_H"] \arrow[dr, "\rho_{H'}"'] & & R \otimes H \\ & R \otimes H' \arrow[ur] & \end{tikzcd}$$

being a commutative diagram, then $H' = H$.

Theorem $R = \left\{ \begin{array}{l} N\text{-Koszul} \\ \text{Noetherian} \end{array} \right.$ AS regular alg., with $\dim_k R_1 = n$.

H = Hopf alg with antipode of finite order coacting on R innerfaithfully with determinant D.

M = matrix corresponding to Nakayama automorphism of R $\left(\begin{array}{l} \text{i.e. } M = I \\ \text{if } R \text{ is CY} \end{array}\right)$

$\mathcal{C}(M) = \{ P \in Mat_n(k) \mid PM = MP \}$ (centralizer of M), subalg. of $Mat_n(k)$

$\mathcal{C}_P(M) = \bigcup_{i \geq 1} \mathcal{C}(M^i)$ (power centralizer of M)

Admittedly we still need to understand this further (*)

If either ① $\mathcal{C}(M)$ commutative, H involutory, D central, or

② $\mathcal{C}_P(M)$ commutative, D^m central for some $m \geq 1$,

then H is cocommutative

when $R = A(\mathbb{E})$ (*) $\Leftrightarrow$ $\mathbb{E}$ being generic.

Corollary Take $R = A(\mathbb{E})$ with $\mathbb{E}$ generic (as specified in paper), H as above. If ① H involutory, D central, or ② D^m central for some $m \geq 1$, then H is cocom.

Next: coactions on higher dimensional regular algs w/ X. Wang & A. Chirvasitu